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Jointly Optimal Precoder and Power Allocation for an Amplify-and-Forward Half-Duplex Relay System

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Abstract– **This paper investigates the optimal precoder design and power allocation between the source and relay for a half-duplex single-relay non-orthogonal amplify-and-forward (NAF) system. Based on the pair-wise error probability (PEP) analysis, an optimal class of 2×2 precoders is first derived for the traditional power allocation scheme, where one-third of the system power is spent at the relay node, while two-thirds are spent at the source node. Different from optimal unitary precoders proposed earlier, the derived class of precoders indicates that the source should spend all its power transmitting a superposition of the symbols in the broadcast phase, while being silent in the cooperative phase, for optimal asymptotic performance. We then further address the problem of jointly optimal precoder and power allocation for the system under consideration. It is shown that the total power should be equally distributed to the source and the relay, and the source should again spend no power during the cooperative phase for the best asymptotic performance. Analytical and simulation results reveal that the proposed precoders not only exploit full cooperative diversity, but also provide significant coding gain over the optimal unitary precoders. For instance, a coding gain of around 1dB can be attained at the practical BER level of 10^{-5} for various modulation schemes.**

Keywords– **Amplify-and-forward, relay channel, error performance, precoder, power allocation.**

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1 Introduction

For wireless channels, diversity is a powerful technique to mitigate the deleterious effects of fading, and as a result, to improve the reliability of the transmission. Among various diversity techniques, antenna diversity has received considerable research interests since it provides diversity without additional cost of increased bandwidth nor transmission time. However, due to practical limitation in putting multiple antennas at the transceivers, attention has been paid to a new way of realizing spatial diversity in a distributed fashion, known as cooperative diversity [1–5]. The strategy is to create a virtual antenna array by deploying relay terminals to assist the source terminal. By appropriate cooperation between the relays and source terminal, it has been widely realized in the literature that full spatial diversity can be achieved in a distributed manner [3, 4, 6].

In general, cooperative schemes can be categorized as decode-and-forward (DF), amplify-and-forward (AF), and compress-and-forward (CF). The AF scheme appears to be of practical interest since relay terminals only need to transmit a scaled version of the signal received from the source terminal, which significantly simplifies the implementation. Among various AF protocols, the half-duplex non-orthogonal AF (NAF) scheme proposed in [4, 7] has been considered to be a general description and superior to other AF schemes, not only from an information-theoretic point of view, but also in terms of the error performance [7, 8].

In the half-duplex NAF protocol, the source terminal is able to continue to transmit the whole time, providing flexibility in designing an effective transmission scheme. Such advantage has been exploited in [9–11] using signal space diversity (SSD) technique, an effective modulation scheme originally proposed in [12–14]. In particular, it was shown in [9, 10] that the application of SSD via a precoder is a simple yet high performance solution to fully exploit cooperative diversity in uncoded NAF wireless relay networks. Based on the pairwise error probability (PEP) analysis, the design of an optimal 2×2 *unitary* precoder for the single-relay scenario was taken into account, first for QPSK constellation in [9]. These results were extended in [10], where optimal real 2×2 *unitary* precoders were provided for any square M-QAM constellation. A main drawback of the proposed scheme in [10] is that this precoder was restricted to the unitary condition. Although such a unitary assumption makes it more feasible for the precoder design, it does not guarantee that the solution is globally optimal. In addition, the studies in [9, 10] were only based on the conventional power allocation scheme in which one-third of the system power is spent at the relay node, while two-thirds are spent at the source node.

In this paper, we consider a general design of 2×2 precoders and power allocation between the source

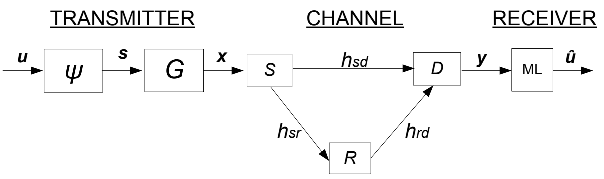

Figure 1. Block diagram of a NAF system using the precoder $\boldsymbol{G}$

and the relay nodes for the half-duplex NAF system, without imposing the unitary condition. Based on the PEP analysis, an asymptotically optimal class of 2×2 precoders is first derived for the conventional power allocation scheme. Different from the optimal unitary precoders proposed in [9, 10], it is shown that to achieve the best asymptotic performance, the source node should convey the superposition of the signals in the broadcast phase, while being silent in the cooperative phase. Then, the jointly optimal precoder and power allocation for such a system is further addressed. It is shown that to optimize the asymptotic error performance, the total transmitted power of the NAF system should be equally divided to the source and the relay, and the source should again spend all its allocated power in the first phase. Analytical and simulation results reveal that the developed precoders not only exploit full cooperative diversity but also offer a remarkable coding gain over the optimal unitary precoders.

This paper is organized as follows. Section 2 describes the structure of the single-relay NAF system using a 2×2 precoder. The PEP analysis is then presented in Section 3. An optimal class of 2×2 precoders for the conventional power allocation scheme is derived in Section 4. In Section 5, a jointly optimal solution for precoder design and power allocation is addressed. Illustrative results are then provided in Section 6 to confirm the analysis. Finally, Section 7 concludes the paper.

2 System Model

A general block diagram of the NAF system using a 2×2 precoder $\boldsymbol{G}$ is shown in Fig. 1. The information sequence $\boldsymbol{u}$ is first divided into groups of $2m_c$ bits. Each group is mapped to a signal $\boldsymbol{s}=[s_1,s_2]^T$ in the complex 2-dimensional (2-D) constellation Ψ. Each component s_i is assumed to be in 1-D constellation Ω, such as QPSK or QAM, of size 2^{m_c}. The symbol $\boldsymbol{s}\in\Psi$ is then rotated by a 2×2 precoder $\boldsymbol{G}$. The rotated symbol $\boldsymbol{x}=[x_1,x_2]^T$ corresponding to a new rotated constellation Ψ_r is given by $\boldsymbol{x}=\boldsymbol{G}\boldsymbol{s}$, where $\boldsymbol{G}$ is a 2×2 rotation matrix with entries $\{g_{ik}\}$, $1\le i,k\le 2$.

Each symbol $\boldsymbol{x}\in\Psi_r$ is sent over the NAF channel via two cooperative phases. In the broadcasting phase, the source S sends the first component of $\boldsymbol{x}$ to both the relay R and the destination D. The received signals at these two nodes can be written respectively as,

$$r_1=\sqrt{E_s}h_{sr}x_1+w_1 \ \text{ and, } \ d_1=\sqrt{E_s}h_{sd}x_1+v_1,$$

where E_s is the transmitted symbol energy; w_1 and v_1 denote the i.i.d. zero-mean circularly Gaussian noise with variance N_o, denoted as $\mathcal{CN}(0,N_o)$, received at R and D in the first phase, respectively; and h_{sr} and h_{sd} denote the S-R and S-D channel gains, also respectively. In the cooperative phase, S sends the second component of $\boldsymbol{x}$ to D, while R sends the symbol received during the broadcast phase to D. The received signal at D for this phase is expressed as,

$$d_2=\sqrt{E_s}h_{sd}x_2+h_{rd}b(\sqrt{E_s}h_{sr}x_1+w_1)+v_2,$$

where h_{rd} is the R-D channel gain and b is the amplification coefficient.

In this paper, similar to [4, 9–11], we assume that all channel gains h_{sr}, h_{rd}, and h_{sd} are i.i.d. $\mathcal{CN}(0,1)$, remain constant during the two cooperative phases, and are perfectly known at D. Furthermore, the relay only has knowledge about the second order statistics of the S-R channel, which makes $b=\sqrt{mE_s/(\eta_1E_s+N_o)}$, where $\eta_i=\sum_{k=1}^{2}\|g_{ik}\|^2$ and m is a parameter that controls the power transmitted at the relay. Note that under this general set-up, the transmitted power at the source and relay are respectively $(\eta_1+\eta_2)E_s$ and mE_s. To satisfy the total power constraint of $3E_s$, one has $\eta_1+\eta_2+m=3$. For a special case of unitary precoders, it is easy to see that $\eta_1=\eta_2=1$ and $m=1$. As a result, $b=\sqrt{E_s/(E_s+N_o)}$ and the transmitted power allocated to the source and relay are fixed at $2E_s$ and E_s, respectively.

By stacking these two phases and whitening the noise components, the matrix model of the received signal at D is written as,

$$\begin{aligned}\boldsymbol{y}&=\sqrt{E_s}\begin{pmatrix} h_{sd} & 0\\ \alpha b h_{rd}h_{sr} & \alpha h_{sd}\end{pmatrix}\begin{pmatrix} x_1\\ x_2\end{pmatrix}+\boldsymbol{n}\\ &=\sqrt{E_s}\boldsymbol{H}\boldsymbol{x}+\boldsymbol{n},\end{aligned} \tag{1}$$

where $\alpha=1/\sqrt{1+b^2\|h_{rd}\|^2}$, and $\boldsymbol{n}\sim\mathcal{CN}(\boldsymbol{0},N_o\boldsymbol{I}_2)$ is a 2×1 complex noise vector. The signal component in (1) can be rewritten as in [9],

$$\boldsymbol{H}\boldsymbol{x}=\boldsymbol{\Sigma}\boldsymbol{X}\boldsymbol{T}\boldsymbol{h}, \tag{2}$$

where,

$$\boldsymbol{\Sigma}=\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix},\ \boldsymbol{X}=\begin{pmatrix}x_1 & 0\\ x_2 & x_1\end{pmatrix},$$

$$\boldsymbol{T}=\begin{pmatrix}1 & 0\\ 0 & bh_{rd}\end{pmatrix},\ \text{ and, } \boldsymbol{h}=\begin{pmatrix}h_{sd}\\ h_{sr}\end{pmatrix}.$$

At the destination D, as depicted in Fig. 1, a maximum likelihood (ML) detector is applied on $\boldsymbol{y}$ to obtain the estimated information sequence $\hat{\boldsymbol{u}}$.

3 Pairwise Error Performance Analysis

In this section, we analyze the pair-wise error probability (PEP) of the system under consideration and derive a bound to this PEP. This bound will be useful in the next sections to first derive an optimal class of precoders for conventional power allocation, and then to end up with a jointly optimal precoder and power allocation scheme. The derivation of PEP follows the one presented in [9].

The PEP is defined as the probability of deciding in favor of $\check{s}$ given that s was transmitted, $\check{s}, s \in \Psi$ and $s \neq \check{s}$. These two signal points correspond to the rotated symbols x and $\check{x}$, i.e. $x = Gs$, and $\check{x} = G\check{s}$, in Ψ_r. Given a perfect channel state information (CSI) at D, the PEP conditioned on H is given by,

$$P(s \to \check{s}|H) = Q\left(\sqrt{\frac{E_s}{2N_o} d^2(x, \check{x}|H)}\right), \quad (3)$$

where $d^2(x, \check{x})$ is the squared Euclidean distance between the two received signals conditioned on H and in the absence of additive white Gaussian noise (AWGN), which is given by,

$$d^2(x, \check{x}|H) = \|HG(s - \check{s})\|^2 = \|H(x - \check{x})\|^2. \quad (4)$$

Applying in (3) the Gaussian probability integral $Q(\gamma) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{\gamma^2}{2\sin^2\theta}\right) d\theta$, the conditional PEP becomes,

$$P(s \to \check{s}|H) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{-1}{c}\|HG(s - \check{s})\|^2\right) d\theta, \quad (5)$$

where $c = \frac{4\sin^2\theta}{\rho}$ with $\rho = \frac{E_s}{N_o}$. Averaging (5) over H results in,

$$P(s \to \check{s}) = \frac{1}{\pi}\int_0^{\pi/2} \Delta(s, \check{s})\, d\theta, \quad (6)$$

where,

$$\Delta(s, \check{s}) = \int \exp\left(\frac{-1}{c}\|HG(s - \check{s})\|^2\right) p_H(H)\, dH. \quad (7)$$

Using the alternative representation of the matrix model shown in (2), the squared Euclidean distance in (4) can be written as,

$$d^2(x, \check{x}|H) = \|\Sigma U T h\|^2 = h^H T^H U^H \Sigma^H \Sigma U T h, \quad (8)$$

where H denotes the Hermitian of a matrix and,

$$U = X - \check{X} = \begin{pmatrix} x_1 - \check{x}_1 & 0 \\ x_2 - \check{x}_2 & x_1 - \check{x}_1 \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ u_2 & u_1 \end{pmatrix}.$$

Then, from (8), by averaging over the circularly distributed Gaussian random vector h, (7) can be simplified to,

$$\begin{aligned}\Delta(s, \check{s}) &= \int\int \exp\left(\frac{-1}{c} d^2(x, \check{x}|H)\right) p_h(h)\, dh\, p_T(T)\, dT \\ &= \int \frac{1}{\det(I_2 + \frac{1}{c} T^H U^H \Sigma^H \Sigma U T)} p_T(T)\, dT, \quad (9)\end{aligned}$$

where we have used the fact that given a complex circularly distributed Gaussian random column vector $z \sim \mathcal{CN}(\mathbf{0}, \Sigma)$ and a Hermitian matrix A, $\mathbb{E}[\exp(-z^H A z)] = 1/\det(I + \Sigma A)$ [9].

Let $y = \|h_{rd}\|^2$, $\epsilon = s - \check{s}$ and $u = (u_1, u_2)^\top$, with $\top$ denoting the transpose operation. Following a similar analysis as in [9], $\Delta(s, \check{s})$ can be reduced to,

$$\begin{aligned}\Delta(s, \check{s}) &= \int_0^\infty \frac{c^2(1 + b^2 y)e^{-y}}{b^2(\|u_1\|^2 + c)^2 y + c(\|u\|^2 + c)}\, dy \\ &= \frac{c^2}{(\|u_1\|^2 + c)^2}\left\{1 + \left(\frac{1}{b^2} - a\right)\exp(a) E_{int}(a)\right\},\end{aligned} \quad (10)$$

where,

$$a = \frac{c}{b^2}\left\{\frac{(\|u\|^2 + c)}{(\|u_1\|^2 + c)^2}\right\},$$

with the exponential integral given as $E_{int}(x) = \int_x^\infty \frac{e^{-u}}{u}\, du$. Thus, the PEP can be obtained by substituting (10) into (6).

To gain more insight into the design of the optimal class of precoders, the Chernoff bound to the PEP is better suited. More specifically, by using the inequality $Q(\sqrt{2x}) < \frac{1}{2}\exp(-x)$, the conditional PEP can now be approximated as,

$$P(s \to \check{s}|H) \approx \frac{1}{2}\exp\left(\frac{-E_s}{4N_o}\|HG(s - \check{s})\|^2\right). \quad (11)$$

Similar to the previous steps, averaging (11) over H gives,

$$P(s \to \check{s}) \approx \frac{1}{2}\Delta_\vartheta(s, \check{s}), \quad (12)$$

where $\Delta_\vartheta(s, \check{s})$ is as shown in (10), with c replaced by $\vartheta = \frac{4N_o}{E_s} = \frac{4}{\rho}$. The next sections address the design of an optimal class of precoders G and an optimal power allocation scheme based on this Chernoff bound.

4 Optimal Class of 2 × 2 Precoders for Conventional Power Allocation

In this section, we consider the design of an optimal class of 2×2 precoders for the conventional power allocation, i.e., the transmitted power at the relay node is E_s, while the source node uses $2E_s$. This power allocation scheme is usually considered in the literature and was applied in [9, 10] for the derivation of the optimal unitary precoders.

From the previous section, it can be seen that the PEP can be reduced by minimizing $\Delta_\vartheta(s, \check{s})$ in (12). By using the approximation to the exponential integral $E_{int}(x) \approx e^{-x}\ln\left(1 + \frac{1}{x}\right)$ [15], $\Delta_\vartheta(s, \check{s})$ can be re-written as,

$$\Delta_\vartheta(s, \check{s}) = \frac{\vartheta^2}{(\|u_1\|^2 + \vartheta)^2}\left\{1 + \left(\frac{1}{b^2} - a_\vartheta\right)\ln\left(1 + \frac{1}{a_\vartheta}\right)\right\}, \quad (13)$$

where,

$$a_\vartheta = \frac{\vartheta}{b^2}\left\{\frac{(\|u\|^2 + \vartheta)}{(\|u_1\|^2 + \vartheta)^2}\right\},$$

with $\|u\|^2 = \|u_1\|^2 + \|u_2\|^2$. Substituting $\vartheta = \frac{4}{\rho}$ and $b = \sqrt{E_s/[(\|g_{11}\|^2 + \|g_{12}\|^2)E_s + N_o]}$, $\Delta_\vartheta(s, \check{s})$ can be asymptotically expressed as,

$$\begin{aligned}\Delta_\vartheta(\boldsymbol{s},\check{\boldsymbol{s}}) &= \frac{16}{\rho^2\|u_1\|^4+O(\rho)} + \\ &\quad \frac{16\rho^2\|u_1\|^4(\|g_{11}\|^2+\|g_{12}\|^2)+O(\rho)}{\rho^4\|u_1\|^8+O(\rho^3)} \times \\ &\quad \ln\left(\frac{\rho^3\|u_1\|^4+O(\rho^2)}{4\rho^2\|\boldsymbol{u}\|^2[\|g_{11}\|^2+\|g_{12}\|^2]+O(\rho)}\right),\end{aligned}$$

where O denotes the big-O notation as $\rho\to\infty$. Ignoring the lower order terms, the above expression can be further approximated as,

$$\begin{aligned}\Delta_\vartheta(\boldsymbol{s},\check{\boldsymbol{s}}) &\approx \frac{16}{\rho^2\|u_1\|^4}+\frac{16(\|g_{11}\|^2+\|g_{12}\|^2)}{\rho^2\|u_1\|^4} \\ &\qquad \times \ln\left(\frac{\rho\|u_1\|^4}{4\|\boldsymbol{u}\|^2[\|g_{11}\|^2+\|g_{12}\|^2]}\right) \\ &= 16\rho^{-2}\ln(\rho)\left\{\frac{\|g_{11}\|^2+\|g_{12}\|^2}{\|u_1\|^4}\right\}+O(\rho^{-2}\ln(\rho)) \\ &\approx 16\rho^{-2}\ln(\rho)\left\{\frac{\|g_{11}\|^2+\|g_{12}\|^2}{\|u_1\|^4}\right\}. \end{aligned} \tag{14}$$

Note that at high SNR, $\Delta_\vartheta(\boldsymbol{s},\check{\boldsymbol{s}})$, and consequently the PEP, do not depend on either g_{21} or g_{22}. To find the optimal precoder, one needs to minimize (14) for a given pair $(\boldsymbol{s},\check{\boldsymbol{s}})$. For a good overall performance, a reasonable approach is to minimize the worst-case PEP.

Let $\boldsymbol{g}_1=(g_{11},g_{12})^\top$. Thus, the optimal precoder can be found by solving the following optimization problem,

$$\min_{\boldsymbol{g}_1}\max_{\boldsymbol{\epsilon}}\left\{\frac{\eta_1}{\|\boldsymbol{g}_1^\top\boldsymbol{\epsilon}\|^4}\right\} \quad \text{s.t.} \;\; 0\le\eta_1\le 2, \tag{15}$$

where recall that $\boldsymbol{\epsilon}=(\epsilon^{(1)},\epsilon^{(2)})^\top=\boldsymbol{s}-\check{\boldsymbol{s}}\neq\boldsymbol{0}$ with $\epsilon^{(i)}=s_i-\check{s}_i$ and $s_i,\check{s}_i\in\Omega$. Since $\boldsymbol{\epsilon}$ depends on the constellation Ω, the optimization problem above is constellation dependent. The solution for the asymptotically optimal class of 2×2 precoders is given in the following theorem.

Theorem 1 *The asymptotically optimal class of 2×2 precoders for $m=1$ is given by,*

$$\boldsymbol{G}^*=\sqrt{2}\begin{pmatrix} g_{11}^* & g_{12}^* \\ 0 & 0\end{pmatrix}, \tag{16}$$

where $\boldsymbol{g}^*=(g_{11}^*,g_{12}^*)^\top$ is the solution of

$$\min_{\boldsymbol{g}}\max_{\boldsymbol{\epsilon}}\left\{\frac{1}{\|\boldsymbol{g}^\top\boldsymbol{\epsilon}\|^4}\right\} \quad \text{s.t.} \;\; \|\boldsymbol{g}\|^2=1. \tag{17}$$

In this case, $\eta_1=2$ and $\eta_2=0$.

Proof: Let $\boldsymbol{g}=\frac{1}{\sqrt{\eta_1}}\boldsymbol{g}_1$. Note that with this normalization, $\|\boldsymbol{g}\|^2=1$. Then, the optimization problem in (15) can be written as,

$$\min_{\boldsymbol{g}_1}\max_{\boldsymbol{\epsilon}}\left\{\frac{1}{\eta_1\|\boldsymbol{g}^\top\boldsymbol{\epsilon}\|^4}\right\} \quad \text{s.t.} \;\; 0\le\eta_1\le 2. \tag{18}$$

Assume that the power transmitted at the source during the broadcast phase is set to r^2, i.e., $\eta_1=r^2$, where $0\le r\le\sqrt{2}$. For this conventional power allocation, the solution to the optimization problem would be,

$$\begin{aligned}\frac{1}{\|\boldsymbol{g}_1^{*\top}\boldsymbol{\epsilon}^*\|^4} &= \frac{1}{r^2\|\boldsymbol{g}^{*\top}\boldsymbol{\epsilon}^*\|^4} \\ &= \frac{1}{r^2}\min_{\boldsymbol{g}}\max_{\boldsymbol{\epsilon}}\left\{\frac{1}{\|\boldsymbol{g}^\top\boldsymbol{\epsilon}\|^4}\right\}, \quad \text{s.t.} \;\; \|\boldsymbol{g}\|^2=1. \end{aligned}\tag{19}$$

This is equivalent to solving (15) over the surface of the sphere $\|\boldsymbol{g}_1\|^2=r^2$. It can be seen from (19) that $\boldsymbol{g}^*$ does not depend on r: for every fixed power allocation at the source during the broadcast phase, the vector $\boldsymbol{g}_1^*$ points in the same direction and has a magnitude of r. Finally, to find the optimal r, one needs to solve,

$$\min_r \frac{1}{\|\boldsymbol{g}^{*\top}\boldsymbol{\epsilon}^*\|^4}\cdot\frac{1}{r^2} \quad \text{s.t.} \;\; 0\le r\le\sqrt{2}. \tag{20}$$

It is easy to see that $r^*=\sqrt{2}$ and thus $\boldsymbol{g}_1^*=\sqrt{2}\boldsymbol{g}^*$, resulting in $\eta_1=2$ and $\eta_2=0$. ■

Different from the optimal unitary precoders in [9, 10], the derived precoder indicates that the source only needs to send the superposition of signals in the first time slot and being silent in the second one. Equivalently, this means that the source and relay transmit in an orthogonal manner. However, different from the orthogonal AF protocol in [4], the proposed transmission scheme achieves full rate, thanks to superposition modulation. It can be verified that full cooperative diversity is still achieved, as far as the asymptotic performance is concerned.

Note that the solution to (17) for real 2×2 precoders when Ω is a square M-QAM constellation was found in [10] and is given by,

$$\boldsymbol{g}^*=(\cos(\theta_M),\sin(\theta_M))^\top, \tag{21}$$

where $\theta_M=\tan^{-1}\left(\frac{1}{\sqrt{M}}\right)$. For the case of complex 2×2 precoder, the optimal precoder for QPSK was provided in [9] as

$$\boldsymbol{g}_{\text{QPSK}}^*=(\cos(\theta_{\text{QPSK}}),\sin(\theta_{\text{QPSK}})\cdot\exp(j\phi_{\text{QPSK}}))^\top, \tag{22}$$

where

$$\theta_{\text{QPSK}}=\sin^{-1}\left(\sqrt{\frac{3-\sqrt{3}}{6}}\right) \tag{23}$$

and

$$\phi_{\text{QPSK}}=\frac{\pi}{12}. \tag{24}$$

In a general case of QAM constellation, the optimal complex solution still remains unanswered. By using computer searching technique, we conjecture that the optimal complex 2×2 precoder for any square M-QAM can be given as:

$$\boldsymbol{g}_c^*=\left(\cos(\theta_M^{(c)}),\sin(\theta_M^{(c)})\cdot\exp\left(j\phi_M^{(c)}\right)\right)^\top, \tag{25}$$

where

$$\theta_M^{(c)}=\tan^{-1}\left(\frac{1}{\sqrt{M-\left(\sqrt{M}-1\right)\left(2-\sqrt{3}\right)}}\right) \tag{26}$$

and

$$\phi_M^{(c)} = \tan^{-1}\left(\frac{1}{\sqrt{3}+2\left(\sqrt{M}-1\right)}\right). \tag{27}$$

Note that when $M = 4$, it is straightforward to see that g_c^* in (25) is the same as g^*_{QPSK} in (21). Certainly, it is interesting to have a rigorous proof regarding the optimality of g_c^* in (25). Such a study, however, deserves a further investigation. By examining the coding gain achieved by real and complex precoders, it is not hard to verify that there is not much difference between the coding gain obtained by the optimal complex precoder in (25) and that achieved by using optimal real precoder in (21).

Given the above results, the next section addresses the jointly optimal design of a 2×2 precoder and power allocation.

5 Jointly Optimal 2×2 Precoder and Power Allocation

In the previous section, the optimal class of 2×2 precoders for $m = 1$ was derived. This corresponds to the traditional case in which the source and relay nodes are allocated the transmitted power of $2E_s$ and E_s, respectively. Apparently, the scenario is just a special case of a general power allocation scheme as discussed earlier. In this section, by relaxing this restriction, we shall investigate the jointly optimal precoder and power allocation scheme, i.e., jointly optimal $\boldsymbol{G}$, η_1, η_2, and m, to further optimize the coding gain.

Recall from the previous sections that the main objective is to minimize $\Delta_\vartheta(\boldsymbol{s}, \check{\boldsymbol{s}})$. Substituting $\vartheta = \frac{4}{\rho}$ and $b = \sqrt{mE_s/(\eta_1 E_s + N_o)}$ in (13), and following similar derivations as in the previous section, $\Delta_\vartheta(\boldsymbol{s}, \check{\boldsymbol{s}})$ can be asymptotically expressed as,

$$\Delta_\vartheta(\boldsymbol{s}, \check{\boldsymbol{s}}) \approx 16\rho^{-2}\ln(\rho)\left\{\frac{\eta_1}{m\|u_1\|^4}\right\}. \tag{28}$$

Note again that this function does not depend on either g_{21} or g_{22}. The optimal class of precoders for the general power allocation can be then found by solving the following problem,

$$\min_{g_1}\max_{\epsilon}\left\{\frac{\eta_1}{m\|\boldsymbol{g}_1^\top \boldsymbol{\epsilon}\|^4}\right\} \quad \text{s.t.} \quad 0 \le m + \eta_1 \le 3. \tag{29}$$

For a given constellation Ω, one has the following theorem regarding the jointly optimal solution for the precoder and power allocation.

Theorem 2 *The asymptotically optimal class of* 2×2 *precoders is given by,*

$$\boldsymbol{G}^* = \sqrt{\frac{3}{2}}\begin{pmatrix} g_{11}^* & g_{12}^* \\ 0 & 0 \end{pmatrix}, \tag{30}$$

where, $\boldsymbol{g}^* = \left(g_{11}^*, g_{12}^*\right)^\top$ *depends on* Ω *and is the solution to,*

$$\min_{g}\max_{\epsilon}\left\{\frac{1}{\|\boldsymbol{g}^\top \boldsymbol{\epsilon}\|^4}\right\} \quad s.t. \quad \|\boldsymbol{g}\|^2 = 1. \tag{31}$$

As a consequence, the optimal power allocation scheme is the one in which transmitted power is poured equally to the source and relay nodes, i.e., $\eta_1 = m = \frac{3}{2}$ *and* $\eta_2 = 0$.

Proof: As before, by replacing $\boldsymbol{g}_1$ with $\sqrt{\eta_1}\boldsymbol{g}$, the problem in (29) can be written as,

$$\min_{g_1}\max_{\epsilon}\left\{\frac{1}{\eta_1 m\|\boldsymbol{g}^\top \boldsymbol{\epsilon}\|^4}\right\} \quad \text{s.t.} \quad 0 \le m + \eta_1 \le 3. \tag{32}$$

By fixing the power spent at the relay to r_m, i.e. $m = r_m$, and following the same argument as in Theorem 1, the optimal power allocation at the source for the broadcast phase is given by, $\eta_1^* = 3 - r_m$. Then, to find the optimal r_m, one needs to solve the following problem,

$$\min_{r_m}\frac{1}{\|\boldsymbol{g}^{*\top}\boldsymbol{\epsilon}^*\|^4}\cdot\frac{1}{r_m(3-r_m)} \quad \text{s.t.} \quad 0 \le r_m \le 3 \tag{33}$$

Observe that $\frac{1}{\|\boldsymbol{g}^{*\top}\boldsymbol{\epsilon}^*\|^4}$ does not depend on r_m. Then by combining with the results from (17), (21), and (25), it can be seen that the problem in (33) is equivalent to find the minimum value of $\frac{1}{r_m(3-r_m)}$. By applying the Cauchy inequality to $\frac{1}{r_m(3-r_m)}$, it is easy to verify that the optimal solution is achieved when $r_m^* = \frac{3}{2}$, and thus $\eta_1^* = m^* = \frac{3}{2}$ and $\eta_2^* = 0$. ■

For the jointly optimal solution, as similar to the traditional power allocation scheme, we can see again that no power is allocated to the source during the cooperative phase. However, different from the previous section, the total power of the system is shown to be equally distributed to the source and the relay.

6 Illustrative Results

In this section, simulation results are provided to confirm the analysis carried out in the previous sections. In all simulations, the bit-error rate (BER) is plotted versus E_b/N_o, where E_b is the energy per information bit. Furthermore, only Gray labelling scheme and square M-QAM constellations are considered. The selected precoders include:

$$\boldsymbol{G}_M = \begin{pmatrix} \cos(\theta_M) & \sin(\theta_M) \\ -\sin(\theta_M) & \cos(\theta_M) \end{pmatrix},$$

which is the optimal real *unitary* precoder given in [10], and

$$\boldsymbol{G}_M^*(r_m) = \sqrt{r_m}\begin{pmatrix} \cos(\theta_M) & \sin(\theta_M) \\ 0 & 0 \end{pmatrix}.$$

According to Theorem 1, $\boldsymbol{G}_M^*(2)$ is the most suitable choice when the traditional power allocation scheme is applied. On the other hand, $\boldsymbol{G}_M^*(3/2)$ is optimal according to Theorem 2. Also note that with $\boldsymbol{G}_M^*(r_m)$, the power allocated at the source in the broadcast phase is $r_m E_s$, while the relay uses $(3 - r_m)E_s$ in the cooperative phase (the source keeps silent during the cooperative phase). For the precoder $\boldsymbol{G}_M$, the source spends a total power of $2E_s$, which is divided equally between the broadcast and cooperative phases, while E_s is allocated to the relay in the cooperative phase.

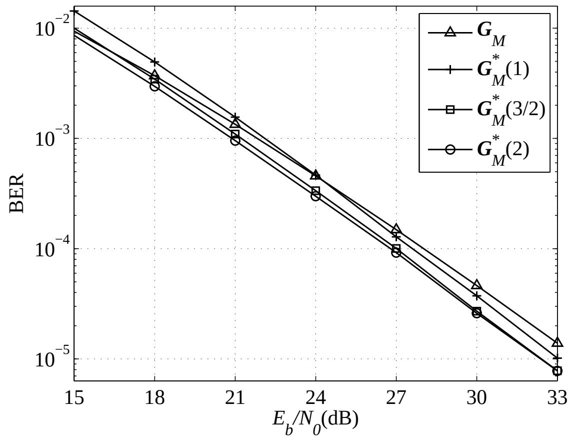

Figure 2. BER performance of the NAF system using 2×2 precoders G_M, $G^*_M(1)$, $G^*_M(3/2)$ and $G^*_M(2)$ for QPSK constellation ($M = 4$).

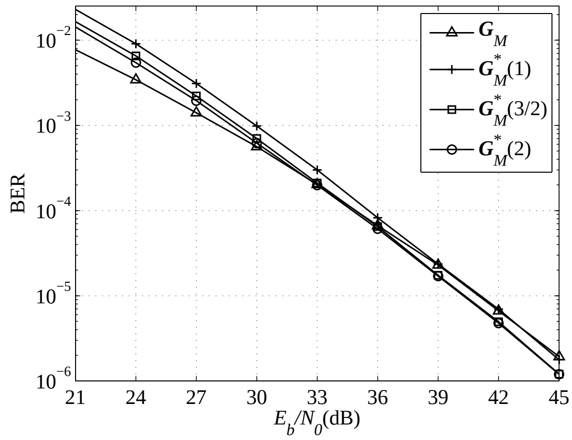

Figure 3. BER performance of the NAF system using 2×2 precoders G_M, $G^*_M(1)$, $G^*_M(3/2)$ and $G^*_M(2)$ for 16-QAM constellation ($M = 16$).

To verify the optimality of the proposed precoders, Fig. 2 shows the BER performances of the NAF system using precoders G_M, $G^*_M(1)$,$G^*_M(3/2)$ and $G^*_M(2)$ for the QPSK constellation ($M = 4$). First, note that at sufficiently high SNRs, the three non-unitary precoders $G^*_M(1)$, $G^*_M(3/2)$ and $G^*_M(2)$ outperform the optimal unitary precoder G_M, especially with $G^*_M(3/2)$ and $G^*_M(2)$. In particular, at the practical BER level of 10^{-5}, the coding gains achieved by $G^*_M(3/2)$ and $G^*_M(2)$ over the optimal unitary rotation are more than 1dB.

It is also observed from Fig. 2 that both $G^*_M(3/2)$ and $G^*_M(2)$ precoders outperform the $G^*_M(1)$ at sufficiently high SNRs. Note that these three precoders all belong to the proposed orthogonal transmission scheme. The $G^*_M(1)$ precoder is clearly suboptimal given that E_s and $2E_s$ are allocated at the source in the broadcast phase and at the relay in the cooperative phase, respectively. Even though $G^*_M(3/2)$ is optimal according to Theorem 2, both $G^*_M(3/2)$ and $G^*_M(2)$ give a similar error performance. It is because the parameter $\frac{1}{r_m(3-r_m)}$ in (33) is equal to 1/2 and 1/2.25 for the traditional and optimal power allocation schemes, respectively, which makes their corresponding coding gains very comparable. A slight advantage of $G^*_M(3/2)$ over $G^*_M(2)$ can be observed at higher SNR ranges. This is because a larger SNR is required for $G^*_M(3/2)$ to be superior than $G^*_M(2)$.

Finally, Fig. 3 presents the same BER performance comparison as in Fig. 2 but for the 16-QAM constellation. It can be seen from this figure that at sufficiently high SNRs, G_M and $G^*_M(1)$ present identical performance, and so do the precoders $G^*_M(3/2)$ and $G^*_M(2)$. The superiority of $G^*_M(3/2)$ over $G^*_M(2)$ can be observed at higher SNRs. More importantly, it is seen that the optimal precoders $G^*_M(3/2)$ and $G^*_M(2)$ outperform again the optimal unitary precoder G_M. In particular, a coding gain of of 0.9dB at the BER level of 10^{-5} can be achieved over the optimal unitary precoder. Note that for $M = 16$, the proposed optimal precoders require higher SNRs to outperform the optimal unitary precoder. Whereas the cross-over occurs at the BER level of 10^{-2} for the QPSK constellation, it happens around 10^{-4} for the 16-QAM. It is reasonable, due to the fact that our analytical analysis in the previous sections concentrates on high SNR regimes.

7 Conclusions

This paper studied an optimal precoder design and power allocation for the NAF system. Based on the worst-case PEP analysis, the optimal class of 2×2 precoders in terms of the asymptotic performance was first derived for the conventional power allocation scheme. In contrast to the optimal unitary precoders, it was shown that the source should spend all its power to transmit a superposition of the symbols in the broadcast phase. In the cooperative phase, only the relay forwards this superposition signal to the destination. Furthermore, by considering a general case of power allocation, it was demonstrated that the total power should be equally divided between the source and relay and the source should spend again all its power in the broadcast phase. Numerical results were provided for various modulation schemes to confirm the optimality of the proposed precoders.

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